The Hamiltonian formalism

We have already encountered the Hamiltonian H in connection with energy conservation in the Lagrangian formulation of mechanics. Let's review the equations we obtained in the case of Cartesian coordinates:

$$H = T + V = \sum_{i=1}^{N} \tfrac{1}{2} m_i \dot{x}_i^2 + V(x_1, \ldots, x_N)$$

We are using "x" for all components of the particle positions to keep the notation simpler. For a 3D system N is a multiple of 3, and $m_1 = m_2 = m_3$ is the mass of one particle with coordinates (x_1, x_2, x_3) etc.

$$P_i = \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i \quad \left(\text{momentum conjugate to } x_i\right)$$

We have N equations of motion, each one a 2nd order differential equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x}$$

$$\Downarrow$$

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} \qquad (i = 1, \ldots, N)$$

In the Hamiltonian formalism we think of H as being a function of the N coordinates (x_i) and the N <u>momenta</u> (P_i), rather than the N velocities $(\dot{x}_i)$. We should therefore express H accordingly:

$$\dot{x}_i = \frac{p_i}{m_i}$$

$$H = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{p_i}{m_i}\right)^2 + V(x_1, \dots, x_N)$$

$$= \sum_{i=1}^{N} \frac{1}{2} \frac{p_i^2}{m_i} + V(x_1, \dots, x_N)$$

With the N x's and N p's treated as independent variables of H, we obtain the following partial derivatives of H:

$$\frac{\partial H}{\partial x_i} = \frac{\partial V}{\partial x_i}, \quad \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i}$$

But $$-\frac{\partial H}{\partial x_i} = \underbrace{-\frac{\partial V}{\partial x_i} = m_i \ddot{x}_i}_{\text{Euler-Lagrange}} = \dot{p}_i$$

and $$\frac{\partial H}{\partial p_i} = \frac{p_i}{m_i} = \dot{x}_i$$

We therefore have 2N anti-symmetrical 1st order differential equations:

$$\left.\begin{aligned} \dot{P}_i &= -\frac{\partial H}{\partial X_i} \\ \dot{X}_i &= +\frac{\partial H}{\partial P_i} \end{aligned}\right\} \quad \begin{array}{l} i = 1, \dots, N \\ 2N \text{ 1st order} \\ \text{equations} \end{array}$$

This nice anti-symmetrical structure is unchanged, we will find, when we use generalized coordinates!

Recall, $P_i = \dfrac{\partial L}{\partial \dot{q}_i}$ (~~for~~ momentum conjugate to gen. coord. q_i)

$$H = \sum_{i=1}^{N} P_i \dot{q}_i - L$$

Now we should think of

H as expressed only in terms of q's, p's, and t (no $\dot{q}$'s).

We do this by using the N equations

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = f_i(q\text{'s}, \dot{q}\text{'s}, t)$$

to solve for each $\dot{q}_i$ in terms of q's, p's, and t.

Let's take partial derivatives of H and see what equations we obtain:

$$\frac{\partial H}{\partial p_j} = \dot{q}_j + \sum_{i=1}^{N} p_i \frac{\partial \dot{q}_i}{\partial p_j} - \frac{\partial L}{\partial p_j}$$

In the two partial derivates on the right we should think of all $\dot{q}$'s as expressed in terms of q's and p's and t, with all three of these treated as independent. ~~We use the~~ But L is defined as a function of independent variables $q, \dot{q}, t$ so we need to use the multi-variable chain rule to perform the indicated derivative:

$$\left.\frac{\partial L}{\partial p_j}\right|_{q\text{'s}} = \sum_{i=1}^{N} \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q\text{'s}} \frac{\partial \dot{q}_i}{\partial p_j}$$

notation for "keep constant"

$$= \sum_{i=1}^{N} p_i \frac{\partial \dot{q}_i}{\partial p_j}$$

But that's exactly the second term in $\frac{\partial H}{\partial p_j}$. Hence:

$$\frac{\partial H}{\partial p_j} = \dot{q}_j$$

Now the other set of equations.
Again we use the multi-variable chain rule being very careful to keep track of which varibles are being held fixed:

$$\left.\frac{\partial H}{\partial q_j}\right|_{p's} = \sum_{i=1}^{N} p_i \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{p's} - \left.\frac{\partial L}{\partial q_j}\right|_{p's}$$

last term:

$$\left.\frac{\partial L}{\partial q_j}\right|_{p's} = \underbrace{\left.\frac{\partial L}{\partial q_j}\right|_{\dot{q}'s} + \sum_{i=1}^{N} \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q's}}_{} \frac{\partial \dot{q}_i}{\partial q_j}$$

we include both sets of var's L is defined in terms of: q's & $\dot{q}$'s

$$= \underset{\text{(Euler-Lagrange)}}{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)} + \underset{\text{(def. of } p_i\text{)}}{\sum_{i=1}^{N} p_i \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{p's}}$$

$$= \dot{p}_j + \sum_{i=1}^{N} p_i \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{p's}$$

$$\Rightarrow \frac{\partial H}{\partial q_j} = -\dot{p}_j$$

This shows the form of the 2N "Hamilton's equations" ~~have exactly~~ is exactly what it was in the cartesian case:

$$\begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \\[2ex] \dot{q}_i = +\dfrac{\partial H}{\partial p_i} \end{cases}$$